Stability and chaos of dynamical systems and interactions

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Models of Human-Environment Interaction
Lecture 3, April 15/29, 2015
Basic types of dynamic mathematical models

\( x(t) \): System state at time \( t \)

\( \Delta x(t) = x(t+1) - x(t) \): System change at time \( t \)

\( \Delta x(t) = f(x,t) \): dynamic system

\( \Delta x(t) = f(x,u,t) \): dynamic system with control variable \( u \)

\( \Delta x(t) = f(x,u_1,u_2,t) \): dynamic game with control variables \( u_1,u_2 \) of two agents 1 and 2

\( \Delta x(t) = f(x,u_1,\ldots,u_n,t) \): agent-based model and social network with control variables \( u_1,\ldots,u_n \) of multiple agents 1,\ldots,n

→ How to select control variables?
General notation:

\( \Delta x \): Change of system variable \( x \in X \) in state set \( X \)
\( \Delta x(t) = dx(t)/dt = \dot{x}(t) \): Change at time \( t \) for time-continuous case
\( \Delta x(t) = x(t+1) - x(t) \): Change at time \( t \) in time-discrete case

General dynamic system: Pair \( (X, F) \) with

- Transformations \( F^t \) in time \( t \in T \) system states \( x \in X \)
- Trajectory of state sequences \( F^1 \circ \ldots \circ F^t \circ x^0 \) from initial point \( x^0 \) result in trajectories in state space \( X \)
Definitions on dynamic systems

Definition: A dynamic system on $X$ is a mapping (flow) $F : X \times T \to X$ with the following properties:

(a) $F(x, 0) = x$ for all $x \in X$ (identity).
(b) $F(F(x, t), s) = F(x, t+s)$ for all $x \in X$ and $t, s \in T$ (semi-group property).

The orbit (trajectory) through $x \in X$ is the point set

$$O_T(x) = \{F(x, t) | t \in T\}$$
Patterns of dynamic systems

Fixed point or equilibrium point of a flow $F : X \times T \to X$ if

$$F(x, t) = x \quad \text{for all} \quad t \in T.$$ 

Periodic trajectory: for the path $F(x, t)$ with $x \in X$ and $t \in T$, a period $p > 0$ exists such that

$$F(x, t + p) = F(x, t) \quad \text{for all} \quad t \in T.$$ 

Differential equation: Let $F : X \to X$ be a mapping on the state set $X$ and

$$\dot{x} = F(x(t))$$

with the initial condition $x(0) = x^0$. Then a solution is given as $x(t) = f(x_0, t)$. 
What is stability?

General sense of stability: “minor disturbances will not be magnified into a major disturbance, but on the contrary, dampened so as to have only a small and disappearing impact” (Ter Borg 1987: 50).

Stability factors: type of systems, disturbances and responses,

→ Highly sensitive responses (overreactions) could affect the structure of a system and thus its stability.

Stability is dealing with a change between qualitatively different systemic conditions:

- from peace to war
- from cold war to post cold war
- from conflict to cooperation
- from environmental destruction to sustainability
Basic characteristics of stability

1. **System** (object, matter) \( Z = (Z_X, Z_F) \): Which system shall be examined for stability (\( Z_X \) is the set of system states \( X \), \( Z_F \) is the set of state transitions \( F \))? 

2. **Quality** (property, attitude, feature, condition, character, criterion) \( S(X) \): Is the quality \( S \) of the system state \( X \) stable (unstable)? 

3. **Disturbance** (event, force, cause, operation, perturbation, error) \( F \): Which disturbance \( F \) leaves the system quality \( S \) stable? 

4. **Time** (duration, period, interval) \( T \): During which time interval is the system quality \( S \) stable against disturbances \( F \)? 

5. **Probability** (expectance, frequency, weight) \( p_S \): What is the probability \( p_S \) that the system quality \( S \) is stable during the time interval \( T \) against the disturbance \( F \)?

The statement of stability \( Stab(Z, S(X), F, T, p_S) \) is defined as “The quality \( S \) of the state \( X \) of the system \( Z \) is preserved during the time \( T \) against the disturbance \( F \) with the probability \( p_S \).”
Stability conditions of dynamic systems

\[ x = x(t) = (x_1(t), \ldots, x_n(t)) \] Vector of system variables

\[ \dot{x} = Ax = 0 \] Differential equation for matrix A

\[ \lambda_i \ (i = 1, \ldots, n) \] Eigenvalues of matrix A correspond to exponents of exponential functions and are solutions of:

\[ \det | A - \lambda I | = 0 \] Characteristic equation

\[ J = \begin{pmatrix} F_{11}(t) & \cdots & F_{1n}(t) \\ \vdots & \ddots & \vdots \\ F_{n1}(t) & \cdots & F_{nn}(t) \end{pmatrix} \] Jacobi matrix: represents the stability conditions for a non-linear differential equation around the equilibrium

\[ F_{ij} = F_{ij}(\bar{x}_1, \ldots, \bar{x}_n) = \frac{\partial F_i}{\partial x_j} \ (i, j = 1, 2) \] Partial derivatives

The dynamic system is stable if there are no positive eigenvalues.
Typical patterns of fixed points

- Branch point
- Node point
- Ray point
- Saddle point
- Spiral/focal point
- Central point
**Fundamentals of chaos theory**

**Chaos:** System behavior which responds sensitive to small changes in system conditions and leads to a dynamics which is erratic and unpredictable. Even simple deterministic equations can show random-like behavior.

**Concept of deterministic chaos** has been developed for the study of simple nonlinear models with complicated dynamics (e.g. Lorenz 1963, May 1976, Grossmann 1977, Feigenbaum 1978, 1983, Ruelle 1980, Haken 1978, 1982)

Characteristics of deterministic chaos (Schuster 1988):

- deterministic chaos denotes the irregular or chaotic motion that is generated by nonlinear systems whose dynamical laws uniquely determine the time evolution of a state of the system from a knowledge of its previous history.

- observed chaotic behavior in time is neither due to external sources of noise ... nor to an infinite number of degrees of freedom ... nor to the uncertainty associated with quantum mechanics.

- actual source of irregularity is the property of the nonlinear system of separating initially close trajectories exponentially fast in a bounded region of phase space.

- it becomes therefore practically impossible to predict the long-term behavior of these systems, because in practice one can only fix their initial conditions with finite accuracy, and errors increase exponentially fast.
One-dimensional quadratic iterative map

Properties of chaos are studied for iterative quadratic mapping \( f_r : T \rightarrow I \) on the unit interval \( T = [0, 1] \) with \( x_t \in T \) is variable \( x \) at the discrete time \( t \).

Quadratic (logistic) function (Verhulst):

\[
x_{t+1} = f_r(x_t) = rx_t(1 - x_t).
\]

Fix points: \( \bar{x} = 0, \bar{x} = (r - 1)/r \)

Stability for:

\[-1 < \frac{df(\bar{x})}{dx_t} = r(1 - 2\bar{x}) < 1\]
Logistic map: periodic oscillation

Source: Mainzer 2007
Logistic map: chaos

Source: Mainzer 2007
Bifurcation diagram of logistic map

\[ x_n = r x_{n-1} (1 - x_{n-1}) \]

For any value of the parameter \( r \) the attractor is shown (source: http://mathworld.wolfram.com/LogisticMap.html)
One-dimensional quadratic iterative (logistic) map

Quadratic (logistic) function (Verhulst):

\[ x_{t+1} = f_r(x_t) = rx_t(1 - x_t). \]

\( r \in [0, 4] \) is the reaction coefficient (order parameter), whose value determines the transition from predictability to chaos.

For small \( r \) the variable \( x_t \) converges to stable fixed points \( x^* = 0 \) for \( r < 1 \) and to \( x^* = (r - 1)/r \) for \( r > 1 \), independent of initial conditions.

- For \( r > 3 \) single fixed point bifurcates into periodic cycles with periods \( 2^n \), corresponding to \( n \)-periodic points \( p = f_n(p) \) of \( n \)-th iterate \( f_n = f(f_{n-1}) \).
- Set of iterates of periodic point, \( \{p, f(p), f_2(p), \ldots, f_n(p)\} \), is an orbit of \( p \).
- For \( n \to \infty \) the chaotic region is reached.
- Above \( r \approx 3.85 \), 3-periodic cycles emerge.
Key terms in chaos theory

**Attractor**: set of system states on which system dynamics $x_t = f_t(x)$ stays for sufficiently large times.

**Dissipative systems**: phase-space volumes are contracted and lengths stretched by time evolution such that orbits on attractor are separated exponentially.

**Strange attractors**: Orbits on this attractor are exponentially separated.

**Ljapunov exponent**: exponential separation of initial values $x_0$ and $x_0 + \varepsilon$ after $t$ iterations (analogous to eigenvalues in stability theory), loss of information:

$$
\lambda(x_0) = \lim_{t \to \infty} \frac{1}{T} \log \left| \frac{df^T}{dx} \right|
$$

$\lambda < 0$: Fix point, $\lambda = 0$: periodic limit cycle, $\lambda > 0$: chaotic behavior

**Fractal** (Mandelbrot): Broken dimension, self similarity across different scales (e.g. coast line))

**Ergodic theory** examines relations between dimensions (number of excited degrees of freedom), entropy (production of information), and characteristic exponents (describing sensitivity to initial conditions).
Stability of ecosystems

Biological systems (ecosystems, populations, organisms): dynamical systems able for self protection and self regulation (including structural changes), to maintain viable states for a sufficient time against a stochastic environment (principle of tolerance)

Many meanings and concepts of stability: one of the most foggy terms in ecology (Grimm/Wissel)

Constance: Systemic properties remain essentially constant

Resilience: after a temporary disturbance the system returns to a reference state or dynamics

Persistence: Inertia (Beharrlichkeit) of an ecological system during a particular period

Resistance: System properties remain essentially unchanged, despite the impact of a disturbance

Elasticity: Speed of return to a reference state or dynamics after a temporary disturbance

Area of attraction: Set of all states from which a reference set of states can be reached after a temporary disturbance
Scarce resources limit the existence and well-being of organisms

→ Supply of scarce resources increases production.

Organism depends on weakest part of resource chain.

→ „Liebig‘s Law of the Minimum“: The productivity depends on the most critical limiting factor (e.g. nutrients, light, water)

Limitation by factors with too high concentration

→ „Law of Tolerance“ (Schellford 1913): Each organism has a tolerance range of environmental conditions for existence.

Viability of a system: e.g. fitness, survivbility, efficiency beyond which tolerance range is threatened
Environmental tolerance and viability

Environmental change

Optimum

V = 0

Vulnerable area

Environmental Factor X

X^- Tolerance

X^+ Environmental change

Viability / Productivity V
Environmental tolerance and viability

Viability $V$

Environmental Factor $X$

Environmental variation

Optimum

$V = 0$

$V > 0$

$V < 0$
Dynamic interaction between multiple agents

Interaction between two or more agents $i = 1, ..., n$ (e.g. individuals, groups, populations, organisations, states), which mutually influence each other.

**Competition:** Struggle for influence on particular variables which are related to the interests of agents.

$$\Delta x_i(t) = F_i(x_1(t), ..., x_n(t)) = F_i(x(t)) \quad (i = 1, ..., n)$$

where $x_i(t) \in X_i$ is a **product** (e.g. economic goods, resources, weapons, values, profits, off-spring, emissions, environmental risks) of agent $i$ at time $t$.

$\Delta x_i(t)$ is the new production or reduction at time $t$. 
Dynamic interaction: two products

\[ \dot{x}(t) = F(x(t), y(t)) \]
\[ \dot{y}(t) = G(x(t), y(t)) \]

Fixed points \[ F(\bar{x}, \bar{y}) = G(\bar{x}, \bar{y}) = 0. \]

Competitive situation [Krabs 1997]:

1. For both agents the production functions decline the more goods have been produced (saturation):
   \[ F_x(x, y) = \frac{\partial F}{\partial x} < 0, \quad G_y(x, y) = \frac{\partial G}{\partial y} < 0, \]

2. The growth of one product weakens the production of another product (mutual distraction, direct competition):
   \[ F_y(x, y) = \frac{\partial F}{\partial y} < 0, \quad G_x(x, y) = \frac{\partial G}{\partial x} < 0, \]
Dynamic interaction: two products

3. The growth of one production factor supports the growth of the other (mutual benefit, symbiosis):

\[ F_y(x, y) = \frac{\partial F}{\partial y} > 0, \quad G_x(x, y) = \frac{\partial G}{\partial x} > 0, \]

4. One product supports and one product weakens the production of the other product (asymmetric distribution of benefits and damages):

\[ F_y(x, y) = \frac{\partial F}{\partial y} < 0, \quad G_x(x, y) = \frac{\partial G}{\partial x} > 0, \]

**Stability conditions:** The product of self-effects is larger than the product of mutual effects

\[ \det J = F_x \cdot G_y - F_y \cdot G_x > 0 \]
Interaction in population models

Population dynamics: Change in population number \( N \) by

\[
\Delta N = F(N) = b \cdot N - d \cdot N = r \cdot N
\]

\( b \): Birth rate (births per capita and year),
\( d \): Death rate / mortality (per capita and year)
\( r = b - d \): effective growth rate

Interaction between \( n \) populations: (linear)

\[
\dot{N}_i = F_i(N_1, \ldots, N_n) = F_i(N) = N_i f_i(N) = N_i \left( a_i + \sum_{j=1}^{n} a_{ij} N_j \right) \quad (i = 1, \ldots, n)
\]

\( a_i \): Growth rate of population \( iN_i \)
\( a_{ij} = \frac{\partial f_i}{\partial N_j} \): Effect of population \( N_j \) on growth \( f_i \) of population \( N_i \)
Competition on scarce resources: superiority

\[
\frac{dN_i}{dt} = r_i N_i \left(1 - \frac{N_i + w_i N_j}{K_i}\right)
\]

- \(K_i\): Carrying capacity of population \(K_i\)
- \(w_i\): Weighting factor for the interspecific competition

\[K_1 > K_2/w_2 \text{ und } K_2 < K_1/w_1\]
Competition on scarce resources: coexistence

\[ K_2 > K_1/w_1 \text{ und } K_1 > K_2/w_2 \]
Competition on scarce resources: relative advantage wins

\[ K_1 < K_2/w_2, \text{und } K_2 < K_1/w_1 \]
Resource competition among populations

Two populations \( N_1 \) and \( N_2 \) depend on same resource \( R = \bar{R} - (\gamma_1 N_1 + \gamma_2 N_2) \) (\( \gamma_1, \gamma_2 > 0 \) resource consumption per capita):

\[
\frac{\dot{N}_i}{N_i} = f_i = b_i R - \alpha_1 = a_i - b_i(\gamma_i N_i + \gamma_j N_j) \quad (i, j = 1, \ldots, n)
\]

\( b_1, b_2 > 0 \) Resource utilization rate, \( \alpha_1, \alpha_2 > 0 \): Death rate, \( a_i = b_i \bar{R} - \alpha_i \)

**Exclusion principle:** for each resource \( R \) only one population can survive for which the ratio \( \alpha_i/b_i \) is lowest.
Competition among species

**Exclusion principle:** in each ecological niche only one population can exist.

→ Each population must find the ecological niche most suitable to its characteristics.

Against competitors populations can only survive if they use the niche and its resources more efficiently.

Intensity of competition increases with niche overlap.

→ Limiting difference between niches which allows coexistence.
Food webs and predator-prey models

**Food webs:** multiple populations in hierarchical predator-prey relations

→ Each species feeds on organisms of lower levels and serves as food for organisms of higher levels.

**Predator-prey models** describe the interaction between prey animals and their natural predator animals.

**Volterra:** Why did after World War I the number of predator animals suddenly increase in the Mediterranean?

Volterra model describes the temporal changes of the number of prey fishes $N_1$ and predator fishes $N_2$ as coupled differential equations with processes directed against each other:

$\Delta N_1 = (\text{growth of prey without predators}) - (\text{prey loss due to predators})$

$\Delta N_2 = (\text{growth of predator through captured prey}) - (\text{death of predator for absent prey})$

**Assumption:** Predators need prey for survival and diminish them proportionate to the density of both populations
Predator-prey models

Predator-prey model (Lotka-Volterra): Interaction between prey animals and their predators

\[
\frac{\dot{N}_1}{N_1} = a_1 - c_1 N_2 \\
\frac{\dot{N}_2}{N_2} = -a_2 + c_2 N_1
\]

- $a_1 > 0$ Net growth rate of prey
- $a_2 > 0$: death rate of predators
- $c_1 > 0$: specific loss of prey from predation
- $c_2 > 0$: specific gain of predators from predation

Fix point:

\[(\bar{N}_1, \bar{N}_2) = (a_2/c_2, a_1/c_1)\]

→ Cyclical orbits around fix point
Predator-prey dynamics
Predator-prey dynamics
Predator-prey model

Cyclic paths run around a fix point, growth of predators follows the growth of preys with time delay.

If prey is strongly diminished, the predator population is also diminished with time delay which gives the prey population time for recovery.

If predators are solely dependent on prey, it cannot diminish the prey population below critical density without endangering its own population.

→ Predator-prey relationship cannot lead to extinction of the prey (stable limit cycles)

Predator-prey cycles are hard to empirically validate in nature.

→ Example: Cycles between hare and lynx (recorded in 19. century by Hudson Bay Company): possible effect of human activities?

→ Ideal typical model not provable in reality
Canadian lynx and snowshoe hare pelt-trading records of the Hudson Bay Company

Source: https://www.math.duke.edu/education/webfeats/Word2HTML/Predator.html
Predator-prey interaction of wolves and sheep

Netlogo model
Spatial agent-based framework of wolves-sheep interaction

Cyclical pattern is broken if one species becomes extinct.
Models of conflict and arms race

Evolutionary models of predator-prey relationships served as models of arms race and conflict:

→ „The complex adaptations and counter-adaptations we see between predators and their prey are testament to their long coexistence and reflect the result of an arms race over evolutionary time.“ (Krebs/Davies 1993)

→ Lanchester model of warfare

→ Richardson model of arms race
Lanchester model of combat

Dynamic models of arms use between two military time dependent variables $X_1$ and $X_2$ of two forces (e.g. armies), with two model versions:

$$
\Delta X_1 = -a_2 X_2 \quad \text{for concentrated fire}
$$
$$
\Delta X_2 = -a_1 X_1
$$

$$
\Delta X_1 = -a_2 X_2 X_1 \quad \text{for distributed fire}
$$
$$
\Delta X_2 = -a_1 X_1 X_2
$$

where $a_i = e_i l_i$ is the product of launch/fire rate $l_i$ and attack efficiency $e_i$ of the attacker $i$ on the other force.
Lanchester model of combat

For both types, the force-exchange ratio becomes

\[ ER = \frac{\Delta X_2}{\Delta X_1} = \frac{a_1}{a_2} \]  
for concentrated fire

\[ ER = \frac{\Delta X_2}{\Delta X_1} = \frac{a_1}{a_2} \]  
for distributed fire

During the dynamic duelling the following variables are conserved (constant):

\[ K(t) = a_1 X_1^2(t) - a_2 X_2^2(t) \]  
square law for concentrated fire

\[ K(t) = a_1 X_1(t) - a_2 X_2(t) \]  
linear law for distributed fire

Win by outnumber
units: 1:1.4 damage rate per unit: 1:1

Win by advantage
units: 1:3:1 damage rate per unit: 1:2

Draw: advantage compensate number
units: 1:2 damage rate per unit: 4:1

Lanchester model of warfare and historical data

Total United States Casualties at Iwo Jima

<table>
<thead>
<tr>
<th></th>
<th>Killed</th>
<th>Missing or Died of Wounds</th>
<th>Wounded</th>
<th>Combat Fatigue</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Marines</td>
<td>5,931</td>
<td>17,272</td>
<td>2,648</td>
<td>25,851</td>
<td></td>
</tr>
<tr>
<td>Navy units:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ships and air units</td>
<td>633</td>
<td>1,158</td>
<td>1,791</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Medical corpsmen</td>
<td>195</td>
<td>529</td>
<td>724</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Seabees</td>
<td>51</td>
<td>218</td>
<td>269</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Doctors and dentists</td>
<td>2</td>
<td>12</td>
<td>14</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Army units in battle</td>
<td>9</td>
<td>28</td>
<td>37</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Grand Totals</td>
<td>6,821</td>
<td>19,217</td>
<td>2,648</td>
<td>28,686</td>
<td></td>
</tr>
</tbody>
</table>

Japanese Casualties at Iwo Jima

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Defense Forces</td>
<td>Prisoners</td>
<td></td>
<td>Killed</td>
<td></td>
</tr>
<tr>
<td>(Estimated)</td>
<td>Marine</td>
<td>216</td>
<td>20,000</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Army</td>
<td>867</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Total</td>
<td>1,083</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From [12, p. 296]

Figure 8.5. Mixed Guerrilla–Conventional Combats

Forces in South Vietnam, Spring 1968

<table>
<thead>
<tr>
<th>Conventional Forces</th>
<th>Guerrilla Forces</th>
</tr>
</thead>
<tbody>
<tr>
<td>American</td>
<td>North Vietnamese</td>
</tr>
<tr>
<td>South Vietnamese:</td>
<td>Viet Cong</td>
</tr>
<tr>
<td>regulars</td>
<td>510,000</td>
</tr>
<tr>
<td>local defense</td>
<td>600,000</td>
</tr>
<tr>
<td>Other allies</td>
<td>500,000</td>
</tr>
<tr>
<td>Total</td>
<td>230,000</td>
</tr>
</tbody>
</table>

Force ratio: $\frac{1,680,000}{280,000} = 6$

Figure 8.6. Comparison of Actual Troop Strength with Theoretical for the Battle of Iwo Jima (Adapted from Engel [3])
Limits of the Lanchester model

1. Model uses homogeneous forces and does not distinguish between different types, vulnerabilities, missions and other characteristics.
2. Weapons locations are fixed.
3. The equations are deterministic and do not consider stochastic influences.
4. Attrition coefficients depend on time and state conditions.
5. On the battlefield a mix of direct and indirect fire is probable, reducing the linear and square law to ideal limit cases.
6. There is a delay between launch time and impact time, due to the fight time of the destruction mechanism to the target.
7. No simple metric exists for aggregating the diverse fighting elements of modern conventional forces.
8. Misapplication of aggregated power scores.
9. Logical simplicity of the equations should not lead to optimistic attitude that conventional warfare could be calculated.
10. Empirical validation and prediction of future results is a problem.
Richardson model of the arms race

- Lewis Fry Richardson (1881-1953): British physicist and atmospheric scientist
- "Arms and Insecurity" (1960): written before WW 2
- Application of differential equations and stability theory to social and political processes.

- Predicting and prevent war by general laws in armament dynamics, common to all nations and comparable to the laws of nature.
- Basic assumptions of the Richardson model: Each country increases its own armament level proportional to the armament of the opponent and reduces it proportional to its own armament. Mutual political perceptions are measured by a constant "grievance" term.
- Richardson extended the equations to the arms races 1909-1914 and 1933-1939, using the military expenditures as variable.
- Foreign policy had a "machine-like quality" and increasing armaments could lead to war breakout, while a constant level of armament corresponds to a stationary state without war.
Richardson model of arms race

\[ \Delta X_1 = k_1 X_1 - a_2 X_2 + g_1 \]
\[ \Delta X_2 = k_2 X_2 - a_1 X_1 + g_2 \]

\( k_i \): defense coefficient (military threat) measures the degree to which \( i \) reacts to the opponent’s armament \( X_j \);  
\( a_i \): fatigue coefficient (economic constraint) takes into account effect of economic constraints, reducing the own armament proportional to \( X_i \);  
\( g_i \): grievance term (political-strategic objectives and perceptions); “attitude of threatening or cooperation” (warlike preparations, international trade).

Equilibrium condition \( \dot{X}_i = 0 \): represents by straight lines in \((X_1, X_2)\)-space, whose intersection point \((X_1^*, X_2^*)\) is “balance of power”

\[ X_i^* = \frac{k_ig_j + a_jg_i}{a_1a_2 - k_1k_2}, \]  \(81\)

provided that \( a_1a_2 - k_1k_2 \neq 0 \). For \( g_1 = g_2 = 0 \) the only equilibrium for all coefficients is the point \((0,0)\).
Stability of the equilibria: eigenvalues $\lambda$ of the matrix

$$A = \begin{pmatrix} a_1 & k_1 \\ k_2 & a_2 \end{pmatrix},$$

are the solutions of the characteristic equation $|A - \lambda I| = 0$. Asymptotic stability is guaranteed for all eigenvalues having negative real parts, requiring $\det(A) > 0$, or the stability index

$$\sigma \equiv a_1a_2 - k_1k_2 > 0$$

If the product of defense coefficients $k_i$ dominates the product of fatigue coefficients $a_i$ ($\sigma < 0$) the arms race escalates exponentially (unstable arms race), while for $\sigma > 0$ the armament approaches the equilibrium asymptotically.
Different dynamic behavior of the Richardson model, as a function of fixed points and stability

<table>
<thead>
<tr>
<th>Case</th>
<th>Stability index</th>
<th>Grievance</th>
<th>Fixed point</th>
<th>Dynamics</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$a_1a_2 - k_1k_2 &gt; 0$</td>
<td>$g_1, g_2 &gt; 0$</td>
<td>Stable (positive quadrant)</td>
<td>Stable arms levels</td>
</tr>
<tr>
<td>2</td>
<td>$a_1a_2 - k_1k_2 &lt; 0$</td>
<td>$g_1, g_2 &gt; 0$</td>
<td>Unstable</td>
<td>Unlimited arms race</td>
</tr>
<tr>
<td>3</td>
<td>$a_1a_2 - k_1k_2 &gt; 0$</td>
<td>$g_1, g_2 &lt; 0$</td>
<td>Stable (negative quadrant)</td>
<td>Mutual complete disarmament</td>
</tr>
<tr>
<td>4</td>
<td>$a_1a_2 - k_1k_2 &lt; 0$</td>
<td>$g_1, g_2 &lt; 0$</td>
<td>Unstable</td>
<td>Arms race or mutual disarmament, depending on initial value</td>
</tr>
</tbody>
</table>

Adapted from: Neuneck 1995
Different dynamic behavior of the Richardson model, as a function of fixed points and equilibria (see previous table)

Adapted from: Neuneck 1995  p. 47
Different cases of the Richardson model

Stable equilibrium

Unstable equilibrium (saddle point)

Source: Bigelow 2003
Richardson arms race model before World War 1

\[ \Delta(U+V) = 0.73 (U+V) - 194 \]

\( U + V \): forces of both coalitions

Source: Richardson 1960
Limits of the Richardson model

1. Describing countries by single entity is simplifying.
2. Variables and parameters difficult to measure in reality.
3. Arms race does not only have quantitative aspects, which could be measured by a single variable, but also qualitative aspects.
4. Richardson parameters fixed, decoupled from strategy and security interests, representing mechanistic and deterministic interaction.
5. Both sides have no complete knowledge of the armament levels.
6. Arms buildup and buildup not continuous but discrete-time process
7. Limits of predictability and complexity.
8. Arms buildup not only an action-reaction process but also bureaucratic and budgetary eigendynamics with self-stimulation.
9. Arms race may create crisis unstable situation, but does no necessarily lead to war.
Extensions and adaptations of the Richardson model

- Using number of weapons or their lethality as a measure
- Derivation of Richardson parameters from strategic aims
- Negative Richardson coefficients
- Nonlinear (economic) constraints
- Application of optimal control theory and differential games
- Discretization of time and difference equations
- The influence of the bureaucratic process
- Self-stimulation versus mutual stimulation
- Stochastic Markov processes
- Arms race and war breakout
Other arms race models

Decision rules and self stimulation (Intriligator)

\[ \dot{x}_1(t) = k_1(x_1^*(t) - x_1(t)) \]
\[ \dot{x}_2(t) = k_2(x_2^*(t) - x_2(t)) \]

\( k_i \): Reaction strength
\( x_i^* \): Target level of military force of country \( i = 1, 2 \)

Chaos in the armament dynamics (Saperstein, Grossman, Meyer-Kress)

\[ \Delta x_t = x_{t+1} - x_t = (x_m - x_t)(-k_{11}(x_t - x_s) + k_{12} y_t), \]
\[ \Delta y_t = y_{t+1} - y_t = (y_m - y_t)(-k_{22}(y_t - y_s) + k_{21} x_t), \]

\( x, y \): Military forces
\( x_m, y_m \): Maximal military forces
\( k_{ij} \): Interaction strengths